# The stability of a stratified flow

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This paper deals theoretically with the problem of the hydrodynamic stability of a stratified flow of a viscous fluid. The primary flow consists of two laminar streams of viscous fluids of different densities flowing in opposite directions between two parallel inclined planes under the action of gravity. The effect of surface tension at the interface of the two fluids is included in the formulation of the problem.

Since instability can be expected to occur at low Reynolds numbers when the inclination is nearly vertical, the solution of the Orr–Sommerfeld equations is developed as a power series in the transverse space co-ordinate. It is shown that for the vertical case, the flow is unstable for all values of the Reynolds number. Surface tension is found to influence both the direction and celerity of the disturbance. Results are also given for inclinations slightly away from the vertical, where small critical Reynolds numbers do exist.

## 1. Introduction

For most parallel flows past fixed boundaries whose stability has been investigated so far, instability occurs at rather large Reynolds numbers. The Orr–Sommerfeld equation (see Lin 1955) governing the stability of such flows must therefore be solved for large values of the Reynolds number, which appears as a parameter in these equations. The asymptotic solutions of the Orr–Sommerfeld equation, appropriate for large Reynolds numbers, have singularities at the critical points where the wave velocity of the disturbance is equal to the velocity of the mean flow. Great care must therefore be exercised in the evaluation of these solutions as a critical point is crossed. These singularities of the solutions are, however, not inherent in the Orr–Sommerfeld equation, and are introduced entirely by the method of solution. For flows which can be expected to be unstable at low Reynolds numbers, the appropriate solutions can be expressed in ascending powers of the Reynolds number or of one of the co-ordinates. Since asymptotic solutions are not needed, the aforementioned singularities do not occur. The study of the stability of flows which can be expected to become unstable at low Reynolds numbers can therefore be carried out by conventional methods. In view of this, it is perhaps somewhat surprising that until recent years problems of hydrodynamic instability at low Reynolds numbers have been neglected by research workers.

The flow whose stability is studied here is a stratified flow of two fluids of equal viscosity but different densities. It is entirely motivated by gravity and its velocity distribution is antisymmetric, with the lighter fluid flowing in a direction opposite to that of the heavier fluid, with a point of inflexion of the velocity profile at the interface. Since a point of inflexion is known to have a destabilizing effect, and since the work of Yih (1954) and of Benjamin (1957) has shown that a free surface also has a destabilizing effect, it can be expected to be unstable at low Reynolds numbers, especially if the slope of the mean flow is steep. The Orr–Sommerfeld equation will therefore be solved for small Reynolds numbers.

The type of flow investigated here is encountered in extraction columns in chemical engineering, and can also be found in closed channels such as tunnels and mine shafts. It can also be expected that the results have some bearing on stratified flows occurring in the oceans and the atmosphere, where layers of warm water or air flow over and counter to colder water or air (i.e. the flow of air after cold fronts).

## 2. The primary flow

The present investigation concerns the stability of a steady laminar stratified flow of an incompressible viscous fluid between two parallel fixed planes. The spacing of the planes is denoted by $2b$. The origin of the co-ordinate system is taken half-way between the planes, with the $X$-axis parallel to the planes. The planes are inclined at an angle $\theta$ with the horizontal (see figure 1).

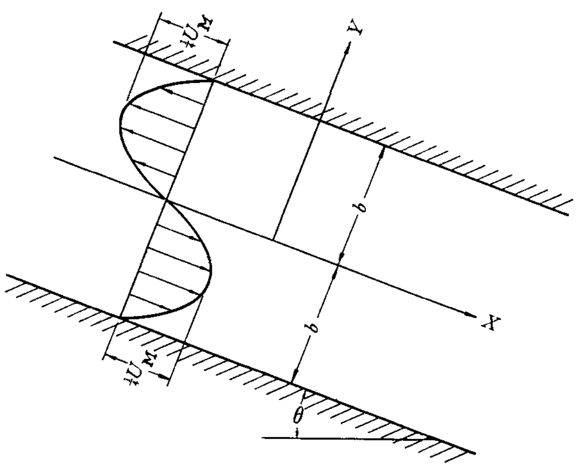


FIGURE 1. Diagram of the primary flow, showing the co-ordinate axes and the velocity profile.

The fluid occupying the region $0 \leqslant Y \leqslant b$, with density $\rho_1$, flows up the inclined plane in the direction of negative $X$. The fluid occupying the region $-b \leqslant Y \leqslant 0$, with density $\rho_2$ greater than $\rho_1$, flows down the plane under the action of gravity. The viscosity $\mu$ of the two fluids are considered equal. Gravity is the sole motivating force for the flow, with the heavier fluid displacing the lighter fluid in a reservoir at $X = +\infty$. The volumetric discharge across the channel is therefore zero.

The Navier–Stokes equations which govern the primary flow are

$$\left.\begin{aligned} 0 &= -P_{1,X} + \rho_1 g \sin\theta + \mu \overline{U}_{1,YY}, \\ 0 &= -P_{1,Y} - \rho_1 g \cos\theta \quad \text{for} \quad 0 \leqslant Y \leqslant b, \end{aligned}\right\} \tag{1}$$

and

$$\left.\begin{aligned} 0 &= -P_{2,X} + \rho_2 g \sin\theta + \mu \overline{U}_{2,YY}, \\ 0 &= -P_{2,Y} - \rho_2 g \cos\theta \quad \text{for} \quad -b \leqslant Y \leqslant 0, \end{aligned}\right\} \tag{2}$$

where $\overline{U}_1$ and $\overline{U}_2$, the components of velocity of the two fluids in the $X$-direction, are functions of $Y$ only, and letter subscripts following commas denote partial differentiation with respect to that quantity. The components of velocity in the $Y$- and $Z$-directions are zero. The condition of continuity is thus automatically satisfied. Pressure is indicated by $P$ and the gravitational acceleration by $g$.

These equations are easily solved for $\overline{U}_1$ and $\overline{U}_2$. It can easily be verified that

$$\overline{U}_1 = U_M\left[-\frac{Y}{b} + \left(\frac{Y}{b}\right)^2\right] \quad (0 \leqslant Y \leqslant b), \tag{3}$$

and

$$\overline{U}_2 = -U_M\left[+\frac{Y}{b} + \left(\frac{Y}{b}\right)^2\right] \quad (-b \leqslant Y \leqslant 0), \tag{4}$$

where

$$U_M = \frac{(\rho_2 - \rho_1)}{4\mu} b^2 g \sin\theta \tag{5}$$

is four times the maximum velocity. The boundary conditions corresponding to (3) and (4) are conditions of no slipping at the fixed boundaries. The condition of zero volumetric discharge is also satisfied by (3) and (4). The pressure gradient in the $X$-direction is

$$P_{1,X} = P_{2,X} = \tfrac{1}{2}(\rho_2 + \rho_1) g \sin\theta.$$

Thus, the $X$-variation of pressure is the same as that of the hydrostatic pressure in a fluid of density $\frac{1}{2}(\rho_2 + \rho_1)$, while the $Y$-variation corresponds to a hydrostatic pressure in the fluids at present under consideration.

## 3. Formulation of the stability problem

### A. *Differential equations*

The basic equations to be satisfied are the Navier–Stokes equations and the continuity equations. Following the usual approach to stability problems, it is assumed that the perturbed velocity and pressure fields are expressible in power series expansions in terms of an amplitude parameter $\epsilon$, assumed small and constant (see Lin 1955, § 1.2). Upon substituting these expansions into the field equations and setting coefficients of powers of $\epsilon$ to zero, it is noted that the equations in $\epsilon^0$ are those already described as governing the primary flow. The equations in $\epsilon^1$ are

$$u^*_{i,T} + \overline{U}_i u^*_{i,X} + \overline{U}_{i,Y} v^* = -\frac{1}{\rho_i} P^*_{i,X} + \frac{\mu}{\rho_1} \nabla^2 u^*_i, \tag{6}$$

$$v^*_{i,T} + \overline{U}_i v^*_{i,X} = -\frac{1}{\rho_i} P^*_{i,Y} + \frac{\mu}{\rho_i} \nabla^2 v^*_i, \tag{7}$$

$$u^*_{i,X} + v^*_{i,Y} = 0, \tag{8}$$

where the disturbance is assumed to be a two-dimensional one. The convention has been adopted that capital letters refer to the primary flow and small letters with an asterisk in the upper right-hand corner refer to the first-order disturbance flow. The equations hold in the upper region when $i = 1$ and in the lower region when $i = 2$. As is customary in stability analyses, attention will be confined solely to the first-order equations, with the assumption that for sufficiently small disturbances the linearized equations can be used without appreciable error.

To facilitate the solution of (6), (7) and (8), it is convenient to put them in dimensionless form. The following dimensionless parameters are introduced

$$\left.\begin{aligned} u_i &= \frac{u_i^*}{U_M}, \quad v_i = \frac{v_i^*}{U_M}, \quad p_i = \frac{p_i^*}{\rho_i U_M^2}, \quad x = \frac{X}{b}, \quad y = \frac{Y}{b}, \\ t &= T\frac{U_M}{b}, \quad U_i = \frac{\overline{U}_i}{U_M}, \quad R_i = \frac{bU_M\rho_i}{\mu}, \quad r = \frac{\rho_2}{\rho_1}. \end{aligned}\right\} \tag{9}$$

Formulating (6), (7) and (8) in terms of these parameters, the following dimensionless equations are obtained

$$u_{i,t} + U_i u_{i,x} + U_{i,y} v_i = -p_{i,x} - \frac{1}{R_i}\nabla^2 u_i, \tag{10}$$

$$v_{i,t} + U_i v_{i,x} = -p_{i,y} + \frac{1}{R_i}\nabla^2 v_i, \tag{11}$$

$$u_{i,x} + v_{i,y} = 0, \tag{12}$$

in which $R_i$ is the Reynolds number based on the half-spacing $b$ and $U_M$. Since the flow is motivated by gravity, it is expected that the Reynolds number and the Froude number $F$ are not independent of one another. Such is indeed the case, for if $F$ is defined by $F^2 = U_M^2/gb$, then from the primary flow (5) provides the relationship

$$F^2 = \tfrac{1}{4}(r-1)R_1 \sin\theta. \tag{13}$$

To reduce the number of equations which are to be solved, it is convenient to introduce stream functions. To simplify solution of (10), (11) and (12), the solutions will be assumed to have exponential time factors. Further, to reduce the partial differential equations to ordinary differential equations, it is assumed that the disturbance may be resolved into Fourier components, which are thus periodic in $x$. A general disturbance then would consist of either a Fourier series or a Fourier integral of such components. To determine the stability of the flow it is sufficient to consider the effect of a single Fourier component of general period.

Only two-dimensional disturbances are considered, since the work of Squire (1933), Yih (1955) and others has shown that the stability or instability of a three-dimensional disturbance can be determined from that of a two-dimensional disturbance at a higher Reynolds number.

The velocity components and the pressure can thus be assumed to have the following forms, in which $f(y)$ and $h(y)$ are introduced as 'stream functions' to satisfy (12) automatically

$$\left.\begin{aligned} u_1 &= f'(y)\, e^{i\alpha(x-ct)}, \\ v_1 &= -i\alpha f(y)\, e^{i\alpha(x-ct)}, \\ p_1 &= q_1(y)\, e^{i\alpha(x-ct)}, \end{aligned}\right\} \tag{14}$$

in $0 \leqslant y \leqslant 1$, and

$$\left.\begin{aligned} u_2 &= h'(y)\, e^{i\alpha(x-ct)}, \\ v_2 &= -i\alpha h(y)\, e^{i\alpha(x-ct)}, \\ p_2 &= q_2(y)\, e^{i\alpha(x-ct)} \end{aligned}\right\} \tag{15}$$

in $-1 \leqslant y \leqslant 0$, where primes denote differentiation with respect to $y$. The dimensionless wave number $\alpha$ is related to the wave length $\lambda$ of the disturbance by $\alpha = 2\pi b/\lambda$. The real part of $c$ represents the dimensionless disturbance celerity, and $\alpha$ times the imaginary part of $c$ gives the dimensionless growth rate of the disturbance.

Upon solving (10), (11), (14) and (15) for the pressure terms, the following are obtained

$$\left.\begin{aligned} q_1 &= \frac{1}{i\alpha R_1}(f''' - \alpha^2 f') + (c - U_1) f' + U_1' f, \\ q_1' &= \frac{1}{i\alpha R_1}(\alpha^2 f'' - \alpha^4 f) + \alpha^2 (c - U_1) f \end{aligned}\right\} \tag{16}$$

in $0 \leqslant y \leqslant 1$, and

$$\left.\begin{aligned} q_2 &= \frac{1}{i\alpha R_2}(h''' - \alpha^2 h') + (c - U_2)\, h' + U_2' h, \\ q_2' &= \frac{1}{i\alpha R_2}(\alpha^2 h'' - \alpha^4 h) + \alpha^2 (c - U_2)\, h \end{aligned}\right\} \tag{17}$$

in $-1 \leqslant y \leqslant 0$. The pressure terms can now be eliminated to obtain

$$f^{\mathrm{iv}} + [-2\alpha^2 + i\alpha R_1(c - U_1)] f'' + [\alpha^4 + i\alpha R_1 U_1'' + i\alpha^3 R_1 (U_1 - c)] f = 0 \tag{18}$$

in $0 \leqslant y \leqslant 1$, and

$$h^{\mathrm{iv}} + [-2\alpha^2 + i\alpha R_2(c - U_2)]\, h'' + [\alpha^4 + i\alpha R_2 U_2'' + i\alpha^3 R_2 (U_2 - c)]\, h = 0 \tag{19}$$

in $-1 \leqslant y \leqslant 0$. These equations are the Orr–Sommerfeld equations. From (3), (4) and (9) the functions $U_1$ and $U_2$ are

$$U_1 = -y + y^2, \quad U_2 = -y - y^2. \tag{20}$$

### B. *Boundary conditions*

Since the differential equations (18) and (19) are two in number and each is of the fourth order, there must be eight boundary conditions imposed to specify the mathematical problem completely. Two conditions are imposed at each of the fixed boundaries, and a total of four are imposed at the interface.

Since the fluids are considered viscous, there must be no slip at the fixed boundaries. Hence, in terms of the stream functions,

$$f(1) = 0, \quad f'(1) = 0, \quad h(-1) = 0, \quad h'(-1) = 0. \tag{21}$$

The interface is assumed to be displaced from the $x$-axis by a small amount which in dimensional form is denoted by $b\eta$. This introduces a further unknown into the problem, which, however, may be readily determined from the kinematical condition that the component of velocity at the interface must equal the time derivative of $\eta$. In what follows, quantities evaluated at the interface are expanded in a Taylors series about $y = 0$, and $\eta$ is considered to be of the same order as the velocity disturbances. In keeping with the previous linearization, only first-order disturbance terms are retained.

At the interface, the velocity components must be continuous; hence

$$f(0) = h(0), \quad f'(0) = h'(0). \tag{22}$$

The kinematic condition at the interface is, in dimensionless terms,

$$v = \eta_{,t} \tag{23}$$

at $y = 0$. Then

$$\eta_{,t} = -i\alpha f(0)\, e^{i\alpha(x-ct)}$$

and upon solving for $\eta$

$$\eta = \frac{1}{c} f(0)\, e^{i\alpha(x-ct)}. \tag{24}$$

The shear stress must also be continuous across the interface. To the first order, the dimensionless shear stress at the interface is given by

$$\tau_{xy} = \frac{1}{R}(U' + u' + v_{,x} + U''\eta). \tag{25}$$

Therefore the boundary condition imposed is, after some simplification,

$$4f(0) + cf''(0) - ch''(0) = 0. \tag{26}$$

The dimensionless normal stress at the interface is

$$\tau_{yy} = -\frac{1}{\rho U_M^2}(P + P_{,Y} b\eta + 2\mu U_M v') - p, \tag{27}$$

where $P$ is the pressure in the primary flow. The condition imposed is that the difference in normal stresses must be equal to the curvature of the interface times $\gamma$, the surface tension. Since $P$ must already be continuous, and since

$$P_{,Y} = -g\rho \cos\theta,$$

the boundary condition becomes

$$q_1(0) - rq_2(0) + \frac{1}{c}\left[\frac{(r-1)\, bg\cos\theta}{U_M^2} + \frac{\alpha^2}{W}\right] f(0) = 0, \tag{28}$$

where $W$ is the Weber number defined by $W = \rho_1 b U_M^2/\gamma$. This can be written entirely in terms of the stream functions by using (16) and (17). The resulting equation is, after further simplification,

$$c[f'''(0) - h'''(0)] + (r-1)\, i\alpha R_1 c[f(0) - cf'(0)] + i\alpha[4\cot\theta + \alpha^2 S] f(0) = 0. \tag{29}$$

The parameter $S$ is defined by

$$S = \frac{R_1}{W} = \frac{\gamma}{\mu U_M}, \tag{30}$$

and represents the ratio of surface tension to viscous forces.

## 4. Solution of the stability problem

A. *General*

For the purpose of solving (18) and (19), much labour can be saved by employing the following notation:

$$\left.\begin{aligned} \phi(y) &= \begin{cases} f(y) & \text{in} \quad 0 \leqslant y \leqslant 1, \\ h(y) & \text{in} \quad -1 \leqslant y \leqslant 0, \end{cases} \\ B &= \alpha^2, \\ A &= \begin{cases} A_1 = i\alpha R_1 & \text{in} \quad 0 \leqslant y \leqslant 1, \\ A_2 = i\alpha R_2 & \text{in} \quad -1 \leqslant y \leqslant 0, \end{cases} \\ \delta &= \begin{cases} +1 & \text{in} \quad 0 \leqslant y \leqslant 1, \\ -1 & \text{in} \quad -1 \leqslant y \leqslant 0. \end{cases} \end{aligned}\right\} \tag{31}$$

Then (18) and (19) can both be solved simultaneously by solving

$$\phi^{\text{iv}} + [-2B + A(c+y-\delta y^2)]\,\phi'' + [B^2 + 2A\delta + AB(-c-y+\delta y^2)]\,\phi = 0. \tag{32}$$

As discussed in the Introduction, instability can be expected to occur at low Reynolds numbers when the inclination $\theta$ approaches 90°. Two techniques at once are suggested as alternate methods of solution. The first, expansion of $\phi$ in ascending powers of the Reynolds number, is certainly the more typical of stability problems and was used by Yih (1954). The alternate approach, expansion of $\phi$ in powers of the co-ordinate $y$, has, to the best of the author's knowledge, been previously employed successfully only by Benjamin (1957) in spite of its early recognition by Kelvin (1887) and others. This method seems more useful in carrying out the solution for $\phi$ to higher powers of the Reynolds number, and conclusions are drawn on the basis of these calculations. Of course, the validity of approximations based on either method can be determined only *a posteriori*.

B. *Expansion in powers of the co-ordinate y*

Following Benjamin (1957), a solution of the form

$$\phi(y) = \sum_{n=0}^{\infty} a_n y^n \tag{33}$$

can be assumed. It is most convenient to expand around the origin since four of the boundary conditions must be satisfied there. Upon substituting (33) into (32) and setting coefficients of $y^n$ to zero, the following recurrence relation is obtained:

$$\begin{aligned} a_n = \frac{(n-4)!}{n!}\{&(n-2)(n-3)(2B-Ac)\,a_{n-2} - (n-4)(n-3)\,Aa_{n-3} \\ &+ [A\delta(n-4)(n-5) - B^2 - 2A\delta + ABc]\,a_{n-4} + ABa_{n-5} - AB\delta a_{n-6}\}. \end{aligned} \tag{34}$$

All of the $a_n$ for $n \geqslant 4$ can now be found from (34) in terms of $a_0$, $a_1$, $a_2$ and $a_3$. Since these four coefficients are arbitrary, four independent solutions of the fourth-order equation are thus obtained. They are (where $a_0$, $a_1$, $a_2$ and $a_3$ have been set equal to 1, 1, 1 and 1/3!, respectively)

$$\begin{aligned}\phi_1(y) = {} & 1+(1/4!)\,(ABc-B^2-2A\delta)\,y^4+(1/5!)\,AB\,y^5\\ & +(1/6!)\,(2A^2c\delta-6AB\delta+3AB^2c-A^2Bc-2B^3)\,y^6\\ & +(1/7!)\,(6A^2\delta+5AB^2-4A^2Bc^2)\,y^7\\ & +(1/8!)\,(-20A^2-3B^4+6AB^3c-4A^2B^2c^2-20AB^2\delta+18A^2Bc\delta\\ & \quad -2A^3c^2\delta+A^3Bc^3-4A^2B)\,y^8\\ & +(1/9!)\,(14AB^3+50A^2B\delta+9A^3Bc^2-22A^2B^2c-16A^3c\delta)\,y^9+\ldots, \end{aligned} \tag{35}$$

$$\begin{aligned}\phi_2(y) = {} & y+(1/5!)\,(ABc-B^2-2A\delta)\,y^5+(1/6!)\,2AB\,y^6\\ & +(1/7!)\,(2A^2c\delta-10AB\delta+3AB^2c-A^2Bc^2-2B^3)\,y^7\\ & +(1/8!)\,(8A^2\delta+8AB^2-6A^2Bc)\,y^8\\ & +(1/9!)\,(-36A^2-3B^4+6AB^3c-36AB^2\delta+30A^2Bc\delta-4A^2B^2c^2\\ & \quad -2A^3c^2\delta-10A^2B+A^3Bc^3)\,y^9\\ & +(1/10!)\,(20AB^3+122A^2B\delta+12A^3Bc^2-30A^2B^2c-20A^3c\delta)\,y^{10}+\ldots, \end{aligned} \tag{36}$$

$$\begin{aligned}\phi_3(y) = {} & y^2+(1/4!)\,(4B-2Ac)\,y^4-(1/5!)\,2A\,y^5\\ & +(1/6!)\,(6B^2-6ABc+2A^2c^2)\,y^6+(1/7!)\,(8A^2c-10AB)\,y^7\\ & +(1/8!)\,(8A^2+16AB\delta-20A^2c\delta+8B^3-12AB^2c+8A^2Bc^2-2A^3c^3)\,y^8\\ & +(1/9!)\,(-36A^2\delta-28AB^2+44A^2Bc-18A^3c^2)\,y^9\\ & +(1/10!)\,(10B^4-20AB^3c+80AB^2\delta+20A^2B^2c^2-164A^2Bc\delta\\ & \quad +64A^2B+76A^3c^2\delta-10A^3Bc^3-56A^3c+2A^4c^4)\,y^{10}+\ldots, \end{aligned} \tag{37}$$

$$\begin{aligned}\phi_4(y) = {} & (1/3!)\,y^3+(1/5!)\,(2B-Ac)\,y^5-(1/6!)\,2Ay^6\\ & +(1/7!)\,(3B^2-3ABc+A^2c^2+4A\delta)\,y^7+(1/8!)\,(6A^2c-8AB)\,y^8\\ & +(1/9!)\,(24AB\delta-22A^2c\delta+10A^2+4B^3-6AB^2c+4A^2Bc^2-A^3c^3)\,y^9\\ & +(1/10!)\,(-80A^2\delta-20AB^2+30A^2Bc-12A^3c^2)\,y^{10}\\ & +(1/11!)\,(160A^2+5B^4-10AB^3c+60AB^2\delta+10A^2B^2c^2+62A^2B\\ & \quad -142A^2Bc\delta+A^4c^4-52A^3c-5A^3Bc^3+62A^3c^2\delta)\,y^{11}\\ & +(1/12!)\,(580A^3c\delta-72A^3Bc^2+90A^2B^2c-80A^3-40AB^3\\ & \quad -64A^2B\delta+20A^4c^3)\,y^{12}+\ldots. \end{aligned} \tag{38}$$

The solutions must converge for all finite $A$, $B$, $c$ and $y$ because of the absence of singularities in the differential equation. An inspection of (35), (36), (37) and (38) indicates that for the range of the parameters considered, the solutions converge rather rapidly. For convenience, a further function $\phi_5$, defined by

$$\phi_5(y) = c\phi_1(y)+\phi_2(y)-\delta\phi_3(y), \tag{39}$$

is also introduced. It is found convenient in later work to use the set of four solutions $\phi_2$, $\phi_3$, $\phi_4$ and $\phi_5$.

The solutions of (18) and (19) are given by linear combinations of the $f_i(y)$ and $h_i(y)$, these being the values of $\phi_i(y)$ when $A$ and $\delta$ are assigned the values $A_1$, 1 and $A_2$, $-1$, respectively.

Satisfaction of boundary conditions (22) and (26) is insured by writing

$$f(y) = a_2 f_2(y) + a_3 f_3(y) + a_4 f_4(y) + a_5 f_5(y), \tag{40}$$

$$h(y) = a_2 h_2(y) + a_3 h_3(y) + a_5 h_5(y) + a_6 h_4(y). \tag{41}$$

To satisfy the remainder of the boundary conditions, it is necessary that

$$\left.\begin{aligned}
0 &= a_2 c(A_1 - A_2) + a_4 i\alpha(4\cot\theta + \alpha^2 S) - a_6,\\
0 &= a_2 f_2(1) + a_3 f_3(1) + a_4 f_4(1) + a_5 f_5(1),\\
0 &= a_2 f_2'(1) + a_3 f_3'(1) + a_4 f_4'(1) + a_5 f_5'(1),\\
0 &= a_2 h_2(-1) + a_3 h_3(-1) + a_5 h_5(-1) + a_6 h_4(-1),\\
0 &= a_2 h_2'(-1) + a_3 h_3'(-1) + a_5 h_5'(-1) + a_6 h_4'(-1).
\end{aligned}\right\} \tag{42}$$

In order for a non-trivial solution of these algebraic equations to exist, the determinant of the coefficients must vanish. Therefore we have

$$0 = \begin{vmatrix}
c(A_1 - A_2) & 0 & 1 & i\alpha(4\cot\theta + \alpha^2 S) & -1\\
f_2(1) & f_3(1) & f_4(1) & f_5(1) & 0\\
f_2'(1) & f_3'(1) & f_4'(1) & f_5'(1) & 0\\
h_2(-1) & h_3(-1) & 0 & h_5(-1) & h_4(-1)\\
h_2'(-1) & h_3'(-1) & 0 & h_5'(-1) & h_4'(-1)
\end{vmatrix}. \tag{43}$$

This determinant may be expanded into a number of two-by-two determinants by a method such as Laplace's expansion. This yields the result

$$\begin{aligned}
0 = H_1F_4 + F_1H_4 - H_2F_6 - F_2H_6 + H_3F_5 + F_3H_5&\\
+ i\alpha(4\cot\theta + \alpha^2 S)(H_2F_3 - F_2H_3) + c(A_2 - A_1)(F_1H_3 - H_1F_3),&
\end{aligned} \tag{44}$$

where the following notation has been introduced:

$$\left.\begin{aligned}
F_1 &= \begin{vmatrix} f_4(1) & f_5(1)\\ f_4'(1) & f_5'(1)\end{vmatrix}, & H_1 &= \begin{vmatrix} h_4(-1) & h_5(-1)\\ h_4'(-1) & h_5'(-1)\end{vmatrix},\\
F_2 &= \begin{vmatrix} f_2(1) & f_4(1)\\ f_2'(1) & f_4'(1)\end{vmatrix}, & H_2 &= \begin{vmatrix} h_2(-1) & h_4(-1)\\ h_2'(-1) & h_4'(-1)\end{vmatrix},\\
F_3 &= \begin{vmatrix} f_3(1) & f_4(1)\\ f_3'(1) & f_4'(1)\end{vmatrix}, & H_3 &= \begin{vmatrix} h_3(-1) & h_4(-1)\\ h_3'(-1) & h_4'(-1)\end{vmatrix},\\
F_4 &= \begin{vmatrix} f_2(1) & f_3(1)\\ f_2'(1) & f_3'(1)\end{vmatrix}, & H_4 &= \begin{vmatrix} h_2(-1) & h_3(-1)\\ h_2'(-1) & h_3'(-1)\end{vmatrix},\\
F_5 &= \begin{vmatrix} f_2(1) & f_5(1)\\ f_2'(1) & f_5'(1)\end{vmatrix}, & H_5 &= \begin{vmatrix} h_2(-1) & h_5(-1)\\ h_2'(-1) & h_5'(-1)\end{vmatrix},\\
F_6 &= \begin{vmatrix} f_3(1) & f_5(1)\\ f_3'(1) & f_5'(1)\end{vmatrix}, & H_6 &= \begin{vmatrix} h_3(-1) & h_5(-1)\\ h_3'(-1) & h_5'(-1)\end{vmatrix}.
\end{aligned}\right\} \tag{45}$$

To complete the determination of the eigenvalue $c$ it becomes necessary now to approximate the $\phi$'s, as obviously the infinite series form for the $\phi$'s must be terminated. It was decided to consider $\alpha$ and $R$ to be of the same order, and to limit the calculation to fifth powers of $\alpha$ and $R$ (and combinations of them). The odd power is introduced by the boundary condition on the normal stresses. No special assumption is made concerning $c$, although it is considered to be of the order of one. Computations based on such an approximation prove straightforward but lengthy. To carry out the next-higher approximation (to the sixth power) would triple or quadruple the amount of work necessary, which is already considerable.

Treating $\alpha$ and $R$ to be of the same order gives results which are valid in a small circle around the origin in the $(\alpha, R)$-plane. Calculations based on expansions to the first order in the Reynolds number as made by Yih (1954) would give results valid in a narrow strip along the $\alpha$-axis in the $(\alpha, T)$-plane. Since the results of most stability analyses (including this one, as will be seen) indicate the onset of instability with small wave-number, it would seem that an approximation valid for a wider range of $R$ is more desirable. This was the main factor in choosing an analysis based on expansion in terms of powers of the co-ordinate $y$.

It is desirable to state the range of $\alpha$ and $R$ for which the approximation is valid. Certainly if $\alpha$ and $R$ are both much less than one, the results are very good. For $\alpha$ and $R$ of the order of one or slightly larger, it is felt that the conclusions are still qualitatively true, since the denominators of the terms in the series increase in a factorial manner while the numerators increase only gradually. However, the prohibitive amount of calculations necessary make any definite statements on the range of validity impossible.

Since the calculations are straightforward, they are not included here. Because in several calculations it was necessary to take the difference between two small numbers of almost equal magnitude, it was considered useful to leave the numbers in fractional form. This also facilitated checking of the results.

When all calculations are carried out, an algebraic equation in integral powers of $c$ is obtained. This is

$$\begin{aligned}
0 = & -320 i\alpha\cot\theta + \tfrac{12}{7}(r+1)\, i\alpha R_1 - \tfrac{320}{3} i\alpha^3\cot\theta + \tfrac{68}{21}(r-1)\,\alpha^2 R_1\cot\theta \\
& -80 i\alpha^3 S - \tfrac{17}{1155}(r^2-1)\,\alpha^2 R_1^2 + \tfrac{86}{105}(r+1)\, i\alpha^3 R_1 + \tfrac{142}{10395}(r^2+1)\, i\alpha^3 R_1^2\cot\theta \\
& -\tfrac{8}{245} r i\alpha^3 R_1^2\cot\theta + \tfrac{388}{315}(r-1)\,\alpha^4 R_1\cot\theta - \tfrac{1856}{105} i\alpha^5\cot\theta \\
& -\tfrac{80}{3} i\alpha^5 S + \tfrac{17}{21}\alpha^4 R_1 S(r-1) \\
& + c\{-1920 - \tfrac{380}{7} i\alpha R_1(r-1) - 1024\alpha^2 - \tfrac{80}{3}(r+1)\,\alpha^2 R_1\cot\theta \\
& \quad + \tfrac{13119}{2310}(r^2+1)\,\alpha^2 R_1^2 - \tfrac{62}{105}\alpha^2 R_1^2 r - \tfrac{608}{21}(r-1)\, i\alpha^3 R_1 - \tfrac{2048}{7}\alpha^4 \\
& \quad - \tfrac{85}{378}(r^2-1)\, i\alpha^3 R_1^2\cot\theta - \tfrac{928}{105}(r+1)\,\alpha^4 R_1\cot\theta - \tfrac{20}{3}\alpha^4 R_1 S(r+1)\} \\
& + c^2\{256(r+1)\, i\alpha R_1 - \tfrac{323}{63}(r^2-1)\,\alpha^2 R_1^2 + \tfrac{1024}{7} i\alpha^3 R_1(r+1) \\
& \quad + \tfrac{6}{7}(r^2+1)\, i\alpha^3 R_1^2\cot\theta + \tfrac{32}{15} i\alpha^3 R_1^2 r\cot\theta\} \\
& + c^3\alpha^2 R_1^2[(r^2+1)\tfrac{181}{14} + 29r]. \qquad (46)
\end{aligned}$$

The root of interest can be computed by assuming $c = c_0 + c_1\alpha + c_2 R + c_3\alpha^2 + \dots$, and substituting this into (46). After this is done, upon separation into real and imaginary parts, the result is

$$c_r = \frac{\alpha^2 R_1(r-1)}{1920}\left\{\frac{2183}{64{,}680}(r+1)R_1 - \tfrac{61}{42}\alpha^2 S - \cot\theta\left[\tfrac{122}{21} - \tfrac{46}{35}\alpha^2\right]\right\}, \tag{47}$$

$$\begin{aligned} c_i = \frac{\alpha}{1920}\bigg\{&\frac{(r+1)R_1}{7}(12 - 2/3\alpha^2) - 16\alpha^2 S(5-\alpha^2) \\ &- \cot\theta\left[64\left(5 - \alpha^2 + \frac{\alpha^4}{21}\right) - \frac{443{,}683}{1{,}164{,}240}(r^2+1)\alpha^2 R_1^2 \right.\\ &\left. + \frac{2787}{17{,}640} r\alpha^2 R_1^2\right] - \cot^2\theta[8/3(r+1)\alpha^2 R_1]\bigg\}. \end{aligned} \tag{48}$$

The growth rate $\sigma$ is given by

$$\sigma = \alpha c_i. \tag{49}$$

It can be argued (see, for instance, Schubauer & Skramstad 1947) that the most likely wave-number to occur is the one with the highest growth rate. This value of $\alpha$ can be determined by setting to zero the derivative of $\sigma$ with respect to $\alpha$. The relation obtained by this is

$$\begin{aligned} 0 = \frac{\alpha}{1920}\bigg\{&\frac{8(r+1)R_1}{7}(3 - \tfrac{1}{3}\alpha^2) - 32S\alpha^2(10 - 3\alpha^2) - \cot\theta\left[64\left(10 - 4\alpha^2 + \frac{6\alpha^4}{21}\right)\right.\\ &\left. - \frac{443{,}683}{291{,}060}(r^2+1)\alpha^2 R_1^2 + \frac{2787}{4410} r\alpha^2 R_1^2\right] - \cot^2\theta\left[\tfrac{32}{3}(r+1)\alpha^2 R_1\right]\bigg\}. \end{aligned} \tag{50}$$

For the case of vertical inclination, the possible roots of (50) are, besides $\alpha = 0$,

$$\alpha_{\text{cr}}^2 = \tfrac{1}{6}[10 + \beta \pm (100 - 88\beta + \beta^2)^{\frac{1}{2}}], \tag{51}$$

where

$$\beta = \frac{(r+1)R_1}{84S}. \tag{52}$$

(In the case of $S = 0$, the roots reduce to 0, 3.) For small $\beta$, (51) may be expanded using the binomial theorem to obtain

$$\alpha_{\text{cr}}^2 = \tfrac{10}{6}\{1 + 0{\cdot}1\beta \pm (1 - 0{\cdot}44\beta - 0{\cdot}092\beta^2)\}, \tag{53}$$

or, for the root of interest,

$$\alpha_{\text{cr}}^2 = \beta\{0{\cdot}9 + 0{\cdot}15\beta\}. \tag{54}$$

## 5. Discussion of results

### A. *The special case of infinite slope*

An inspection of (48) and the corresponding figures 2, 3 and 4 reveals that for the case of vertical inclination and zero surface tension, every value of $\alpha$ makes $c_i$ zero for zero Reynolds number. Hence the $\alpha$-axis is the neutral stability curve. (The analysis does not hold true for large values of $\alpha$, however, it can be argued on physical grounds that in the absence of surface tension there is no restoring force when the slope is infinite.) Surface tension has a stabilizing effect (figures 2, 4) and reduces the range of $\alpha$ for which instability occurs at zero Reynolds

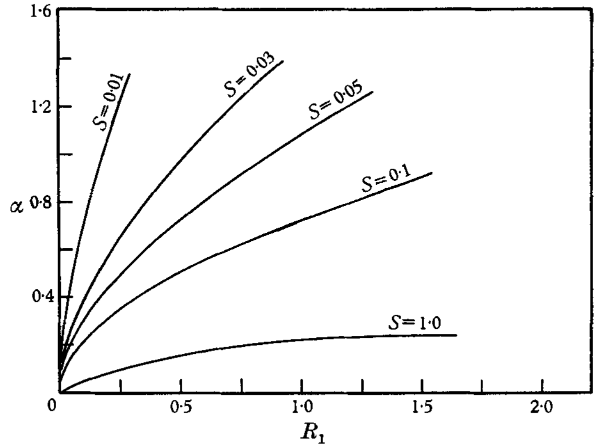


FIGURE 2. Curves of neutral stability for flow between vertical walls with various values of the surface tension parameter $S$. $\sigma = 0$, $r = 1{\cdot}2$, $\theta = 90°$.

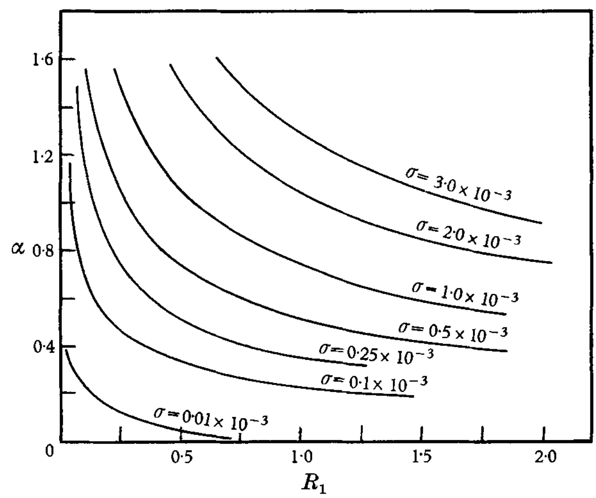


FIGURE 3. The rate of amplification of waves of various wave-numbers for the vertical case. $S = 0$, $r = 1{\cdot}2$, $\theta = 90°$.

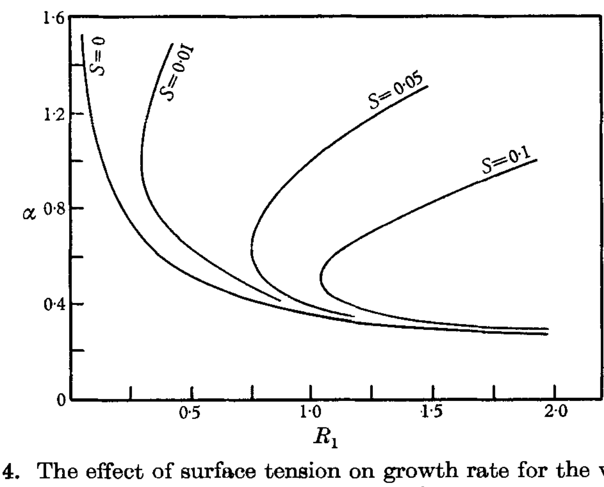


FIGURE 4. The effect of surface tension on growth rate for the vertical case. $\sigma = 0{\cdot}25 \times 10^{-3}$, $r = 1{\cdot}2$, $\theta = 90°$.

number to the point $\alpha = 0$, hence it can never completely prevent instability. From a physical point of view, the effect of surface tension decreases with curvature. From a mathematical point of view, $S$ is always accompanied by a curvature term ($\alpha^2$), so for very small wave-numbers the stabilizing force would be small.

The shape of the growth rate curves of figure 4 is very similar to the neutral stability curve shown by Yih (1954). It might be speculated that the disagreement between Yih's and Benjamin's (1957) results could be due to the difference

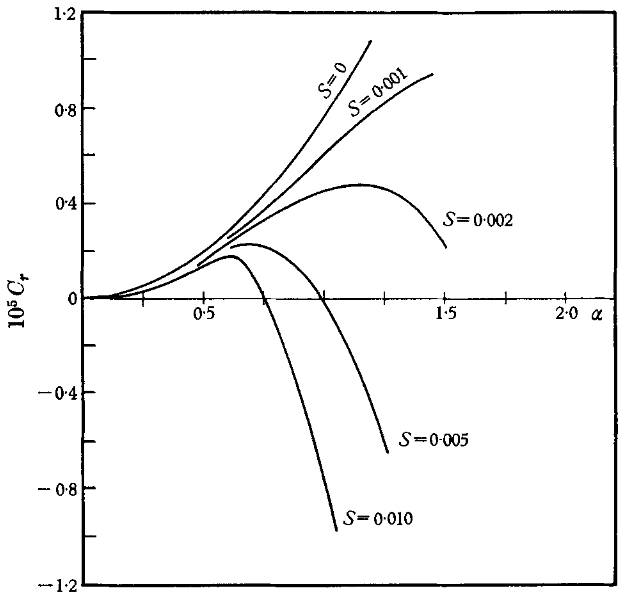


FIGURE 5. The effect of surface tension on the celerity of the disturbance for the vertical case. $R = 1{\cdot}0$, $r = 1{\cdot}2$, $\theta = 90°$.

in the approximations which might yield different members of the same family of curves. Benjamin's result would be the true neutral stability curve, while Yih's would be a curve of constant (small) growth rate.

It is noted from (9) and (47) that the dimensional celerity of the disturbance is directly dependent on the difference in densities of the two fluids and the velocity of the primary flow. A small Reynolds number implies either large viscosity or small difference in densities, hence the speed of propagation of the disturbance is small for the approximation considered. The celerity is also small for the case of small wave-numbers. The disturbance anticipated by the analysis is then in effect almost a standing wave, in the sense that it does not propagate, at least not very quickly.

It is seen from (47) and figure 5 that the surface tension could make the celerity negative, hence disturbances would travel uphill. This effect is perhaps unexpected, but not unreasonable. Such results have in fact been noted in experiments conducted by William M. Sangster of the University of Iowa, for layers of different thicknesses.

If (47) is contrasted with the corresponding results of the stability analyses of Yih and Benjamin, it is seen that the celerities represented in their analyses have values of the same order as the primary flow for very small Reynolds and wave-numbers, while the celerities presented here are many orders smaller than the velocity of the primary flow.

The direct dependence of the disturbance celerity on the difference in densities of the two fluids may seem surprising when compared with previous results (Lamb 1932, p. 370) for gravity waves. The reason for this difference is that the present waves are predominantly influenced by viscosity rather than gravity.

The fact that the primary flow velocity is zero at the interface may explain some aspects of the disturbance celerity. Although a free surface is generally regarded as a destabilizing agent, this effect is somewhat diminished in the present problem due to the counterflow. If the free surface were replaced by a rigid boundary, the problem would be the familiar one of plane Poiseuille flow, for which instability is known to occur for Reynolds numbers many orders higher than contemplated in the present analysis (see, for example, Lin 1955, p. 28).

Another interesting feature is that (24) shows that the disturbance amplitude is inversely proportional to $c$. For small $c$ then, even if the $y$ velocity component is small, the amplitude of the disturbance may be considerably larger in comparison.

Naturally, the onset of instability at such low Reynolds numbers does not mean the onset of turbulence, but only the onset of waves at the free surface. The analysis performed here is, of course, based on the assumption that the two fluids do not mix across the interface.

### B. *Other slopes*

The results are not as informative for values of $\theta$ other than 90°. As the inclination becomes even slightly less steep, the $\theta$ terms in (48) predominate and exert a strong stabilizing influence. For $\theta$ more than a degree of two away from vertical, the results probably are not sufficiently accurate to predict instability, and higher-order approximations are necessitated. Neutral stability curves are shown in figure 6 for various values of the parameter $S$ for the case $\theta = 89{\cdot}5°$. The curve for $S = 0$ bends towards the $\alpha$-axis and predicts instability again for all small Reynolds numbers. It is believed, however, that this is not a reliable result, and that if higher-order terms were present the curve would be almost vertical for small $\alpha$, and then would go in the positive $R_1$ direction, as do the curves for non-zero values of $S$. This same effect is shown in figure 7. The results shown in figure 8, with a non-zero value of $S$, are more reliable in regard to the shape of the curve, and show that for inclinations other than the vertical, critical Reynolds numbers do exist. It seems reasonable that the critical Reynolds number should occur at $\alpha = 0$. The greater surface tension effect and greater dissipation of the disturbance energy at higher wave-numbers would both tend to stabilize the flow. Such tendencies were shown for the case of vertical inclination. A definite mathematical answer to this matter would require a higher-order approximation.

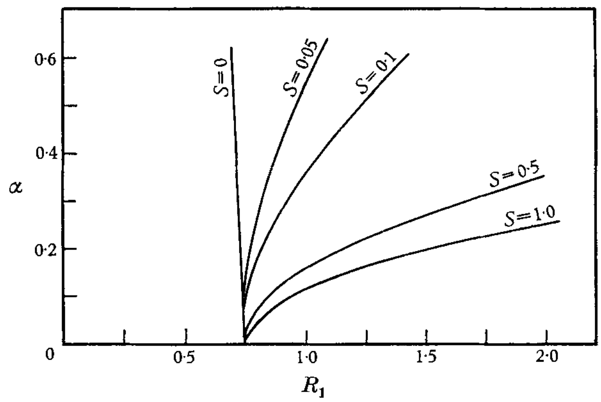


FIGURE 6. Neutral stability curves for the case $\theta = 89{\cdot}5°$, for various values of the surface tension parameter $S$. $\sigma = 0$, $r = 1{\cdot}2$.

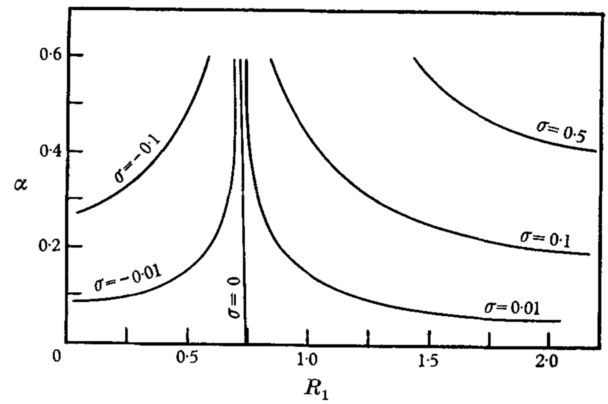


FIGURE 7. The rate of amplification of waves of various wave-numbers for the case $\theta = 89{\cdot}5°$, $S = 0$, $r = 1{\cdot}2$.

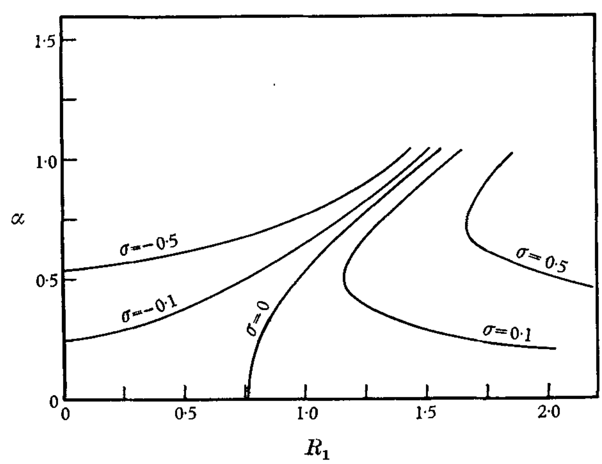


FIGURE 8. The rate of amplification of waves of various wave-numbers for the case $\theta = 89{\cdot}5°$, $S = 0{\cdot}05$, $r = 1{\cdot}2$.

## 6. Conclusions

From the results presented, the following conclusions can be drawn:

(*a*) For the case of vertical inclination, there are values of $\alpha$ for which the flow is unstable at all values of the Reynolds number. The celerity of such disturbances is small, but the effect on the interface can be large.

(*b*) The stabilizing effect of surface tension is shown, although surface tension can never induce complete stability, since it has no effect at zero wave-number.

(*c*) For slopes other than the vertical, a critical Reynolds number exists. The value of the critical Reynolds number probably is the value of $R_1$ which occurs at $\alpha = 0$, although a more thorough investigation is needed in order that a definite statement can be made in this regard.

(*d*) The presence of the interface is responsible for instability at low Reynolds numbers at steep slopes. However, the layer in counterflow does contribute a stabilizing influence.

(*e*) The critical wave number has been found and is given in equation (54). Specific results for the (complex) phase velocity are given in equations (47) and (48).

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